University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

Problems 31-36 afford an opportunity for working with the EM fields generated by a relativistic point particle.

31.
At $t = 0$ at the origin of a spherical polar coordinate system in the lab, a point particle of charge q has velocity βc directed along the $\hat{z}$ (north polar) axis. It has been moving with that constant velocity for a long time.
(a.)
Starting with the Coulomb field in the particle's rest frame, and using the rules for relativistic transformation of EM fields, show that the electric field observed in the lab at $t = 0$ and $\vec{r} = (r, \theta, \phi)$ is

$$4\pi\epsilon_0 \vec{E} = \hat{r}\frac{q}{r^2}\frac{\gamma}{(\gamma^2\cos^2\theta + \sin^2\theta)^{3/2}} \ ,$$

where as usual $\gamma = 1/\sqrt{1-\beta^2}$.
(b.)
Show that this result is equivalent to Griffiths Eq. (10.68).

32.
Griffiths Problem 10.9(b).

33.
The general expression for the electromagnetic fields arising from a point particle of charge q moving with velocity $\vec{\beta}c$ and acceleration $\dot{\vec{\beta}}c$ is

$$\vec{E} = \vec{E}_{\rm v} + \vec{E}_{\rm a} \ \text{ with}$$

$$\vec{E}_{\rm v} = \frac{q}{4\pi\epsilon_0}\Big\{\frac{1}{\mathcal{R}^2}\frac{(\hat{\mathcal{R}}-\vec{\beta})(1-\beta^2)}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\Big\}_{\rm ret}$$

$$c\vec{B} = \big\{\hat{\mathcal{R}}\times\vec{E}\big\}_{\rm ret} \ ,$$

where $\vec{E}_{\rm v}$ is the *velocity field*, and the *acceleration field* $\vec{E}_{\rm a}$ is given in a later problem. Here $\vec{r}$ is a vector from the origin to the observer, $\vec{w}(t)$ is a vector from the origin to the particle, $\vec{\mathcal{R}} \equiv \vec{r} - \vec{w}$, and the subscript "ret" means that quantities are to be evaluated at time $t_{\rm ret} = t - \mathcal{R}/c$.

Assume that $\vec{\beta}$ lies in the z direction and is a constant, so that the acceleration field vanishes. As usual $\theta = \cos^{-1}\hat{z}\cdot\hat{r}$. Choose the origin of coordinates to be the position of the particle at $t = 0$. At that time, show that...
(a.)

$$-ct_{\rm ret} = \gamma\big(\gamma\beta z + \sqrt{(\gamma\beta z)^2 + r^2}\big) \ ;$$

(b.)

$$\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta}) = r\sqrt{1-\beta^2\sin^2\theta} \ ;$$

(c.)

$$\vec{r} = \mathcal{R}(\hat{\mathcal{R}} - \vec{\beta}) \ .$$

34.
Under the conditions of the previous problem, and using the tools developed there, show that $\vec{E}_{\rm v}$ is equivalent to Griffiths Eq. (10.68).

35.
Liénard's equation for the Poynting vector

$$\vec{S}_{\rm a} = \frac{1}{\mu_0}\vec{E}_{\rm a}\times\vec{B}_{\rm a}$$

arising from acceleration of a point particle of charge q is

$$\vec{S}_{\rm a} = (\frac{q}{4\pi\epsilon_0})^2\frac{\epsilon_0}{c}\Big\{\frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\big[\frac{\hat{\mathcal{R}}\times[(\hat{\mathcal{R}}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\big]^2\Big\}_{\rm ret} .$$

(a.)
Show that Liénard's equation follows directly from the electric and magnetic fields arising from acceleration of a point particle, using the *acceleration fields*

$$\vec{E}_{\rm a} = \frac{q}{4\pi\epsilon_0}\frac{1}{c}\Big\{\frac{1}{\mathcal{R}}\frac{\hat{\mathcal{R}}\times[(\hat{\mathcal{R}}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\Big\}_{\rm ret}$$

$$c\vec{B}_{\rm a} = \big\{\hat{\mathcal{R}}\times\vec{E}_{\rm a}\big\}_{\rm ret} \ .$$

(b.)
Suppose that the particle is in uniform motion around a circle of radius b in the plane $z = 0$ centered at the origin. The motion is ultrarelativistic, *i.e.* $(1-\beta^2)^{-1/2} \gg 1$. To lowest order, calculate the radiated power per unit area observed at $(0, 0, z)$, where $z \gg b$.
(c.)
Is $\hat{z}$ a direction in which the power radiated per unit solid angle is near the maximum for this motion? Explain.

36.
As an intermediate step in the derivation of the velocity and acceleration fields $\vec{E}_{\rm v}$ and $\vec{E}_{\rm a}$, in class we derived the expression

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{1-\hat{\mathcal{R}}\cdot\vec{\beta}}\left[\frac{\hat{\mathcal{R}}}{\mathcal{R}^2} + \frac{d}{c\,dt}\frac{\hat{\mathcal{R}}-\vec{\beta}}{\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta})}\right]\right\}_{\rm ret}$$

where the subscript "ret" means that the differentiation should be done first, and afterward all time-dependent quantities should be evaluated at time $t_{\rm ret} = t - \mathcal{R}/c$.

Define $\dot{\vec{\beta}} \equiv d\vec{\beta}/dt$. Use two relations worked out in class:

$$\frac{d\mathcal{R}}{c\,dt} = -\hat{\mathcal{R}}\cdot\vec{\beta}$$

$$\frac{d\hat{\mathcal{R}}}{c\,dt} = \frac{\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})}{\mathcal{R}}\ .$$

With these tools, finish the derivation to obtain $\vec{E}_{\rm v}$ (as given in an earlier problem) and $\vec{E}_{\rm a}$ (as given in the previous problem).